A microscopic derivation of Gibbs measures for the 1D focusing cubic nonlinear Schrödinger equation

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Nonlinear Schrödinger Equation

- We will consider the following form of the NLS

$$\begin{cases} i\partial_t\varphi = (-\Delta+\kappa)\varphi + (w * |\varphi|^2)(x)\varphi(x) \\ \varphi_0 \in H^s(\mathbb{T}), \end{cases}$$

where we take $(x,t) \in (\mathbb{T},\mathbb{R})$.

$$H^s(\mathbb{T}) := \{f : (1+|k|^2)^{s/2}\hat{f}(k) \in \ell^2(\mathbb{Z})\}.$$

- Throughout this talk, we will take
  1. $w \in L^1(\mathbb{T};\mathbb{R})$ to be even with no positivity assumptions on w or $\hat{w}$: focusing Hartree equation
  2. $w = -\delta$: focusing local NLS
- We have two invariant quantities associated with the NLS:

$$M(\varphi) := \int dx\, |\varphi(x)|^2,$$

$$H(\varphi) := \underbrace{\int dx\, |\nabla\varphi(x)|^2 + \kappa|\varphi(x)|^2}_{H_0} + \underbrace{\frac{1}{2}\int dx\, dy\, |\varphi(x)|^2 w(x-y)|\varphi(y)|^2}_{\mathcal{W}}.$$

Gibbs Measures

- We consider a *Gibbs measure* defined heuristically as

$$d\mathbb{P}_{Gibbs} := \frac{1}{z} e^{-H(\varphi)} d\varphi.$$

- For a positive, even L^1 interaction potential, we can write this as

$$\frac{1}{z} e^{-\mathcal{W}} d\mu,$$

  where $d\mu$ is the *Wiener measure*.
- A function in the support of $d\mu$ is given by the random Fourier series

$$\varphi^\omega(x) \equiv \varphi(x) = \sum_{k \in \mathbb{N}} \frac{g_k(\omega)}{\sqrt{\lambda_k}} e^{2\pi i k x},$$

  with g_k gaussian i.i.d. random variables. This is in $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$.
- Gibbs measure is invariant under the flow and for any function in its support, NLS is GWP (Bourgain '94).
- For a negative interaction potential, we introduce a truncation in the mass. Let $f \in C_0^\infty(\mathbb{R})$. Then we consider

$$d\mathbb{P}^f_{Gibbs} := \frac{1}{z} e^{-\mathcal{W}} f(\|\varphi\|^2_{L^2}) d\mu.$$

Many-Body Framework

- Work with n-identical *bosons* – so we need wavefunctions $u \in L^2(\mathbb{R}^n)$ satisfying
$$u(x_1, \ldots, x_n) = u(x_{\pi(1)}, \ldots, x_{\pi(n)})$$
for any $\pi \in S_n$. Let $\mathfrak{h} := L^2(\mathbb{T}; \mathbb{C})$.
- Denote by $\mathfrak{h}^{(p)}$ the symmetric subspace of $\mathfrak{h}^{\otimes_p}$ – *p-body bosonic space*.
- Given a *mass* $m > 0$ and a coupling constant coupling constant $\lambda > 0$, the *n-body Hamiltonian* is defined as
$$H^{(n)} := \frac{1}{m} \sum_{i=1}^{n} (-\Delta_i + \kappa) + \lambda \sum_{i<j=1}^{n} w(x_i - x_j).$$
- Writing $\tau = m$ and $\lambda = \frac{1}{\tau^2}$, we have
$$H_{\tau}^{(n)} = \frac{1}{\tau} \sum_{i=1}^{n} (-\Delta_i + \kappa) + \frac{1}{\tau^2} \sum_{i<j=1}^{n} w(x_i - x_j).$$

Correlation Functions

- The *grand canonical ensemble* is given by the sequence $(\rho_{\tau,n})_n$

$$\rho_{\tau,n} := \frac{1}{Z_\tau} e^{-H_\tau^{(n)}} f\left(\frac{n}{\tau}\right), \quad Z_\tau := \sum_{n\in\mathbb{N}} \mathrm{Tr}_{\mathfrak{h}^{(n)}} e^{-H_\tau^{(n)}} f\left(\frac{n}{\tau}\right),$$

$$Z_{\tau,0} := \sum_{n\in\mathbb{N}} \mathrm{Tr}_{\mathfrak{h}^{(n)}} e^{-H_{\tau,0}^{(n)}} f\left(\frac{n}{\tau}\right),$$

  where $H_{\tau,0}^{(n)}$ corresponds to taking $w = 0$.
- For $p \in \mathbb{N}$, we define the *p-particle reduced density matrix $\gamma_{\tau,p}$* via its operator kernel

$$\gamma_{\tau,p} := \sum_{n\geq p} \frac{p!}{(n-p)!} \mathrm{Tr}_{p+1,\dots,n}(\rho_n)$$

- Given a random variable $X = X(\omega)$, define the *classical Gibbs state* as

$$\rho(X) := \frac{\int d\mu\, X e^{-\mathcal{W}} f(\|\varphi\|_{L^2}^2)}{\int d\mu\, e^{-\mathcal{W}}} = \mathbb{E}_{\mathbb{P}_{Gibbs}}(X).$$

Result

- We define the *classical p-particle correlation functions* through its integral kernel

$$(\gamma_p) := \rho(\overline{\varphi}(y_1)\dots\overline{\varphi}(y_p)\varphi(x_1)\dots\varphi(x_p)).$$

$\rightarrow \mathbb{P}_{Gibbs}$ is completely determined by $(\gamma_p)_p$.

Theorem (R-Sohinger '22)

Let $w \in L^\infty$ be real valued and even. Given $p \in \{1, 2, \dots\}$, we have

$$\lim_{\tau\to\infty} \|\gamma_{\tau,p} - \gamma_p\|_{L^1} \to 0.$$

Moreover

$$\lim_{\tau\to\infty} \frac{Z_\tau}{Z_{\tau,0}} = z.$$

- We can generalise this result to L^1 potentials and the negative δ function.

Placing the Result in the Literature

- Although this is the first known result for focusing interactions, the problem is well studied for defocusing regimes:
    1. **Lewin-Nam-Rougerie ('15)** 1D results using variational method. Non-translation invariant interaction in $d = 2, 3$.
    2. **Fröhlich-Knowles-Schlein-Sohinger ('17)** Bounded potentials in $d = 1, 2$ with modified Gibbs state. New proof of $d = 1$.
    3. **Sohinger ('19)** Extension of above results to optimal $w \in L^q$.
    4. **L-N-R ('18)** $1D$ non-periodic subharmonic trapping potential.
    5. **L-N-R ('18)** $2D$ smooth interaction without modified Gibbs state.
    6. **L-N-R ('20)** Extension to $d = 3$.
    7. **F-K-S-S ('18)** time-dependent problem for $1D$.
    8. **F-K-S-S ('22)** ϕ_2^4 Euclidean field theory for potential with contracting range.